An EXAMPLE PROOF USING THE WELL-ORDERING PRINCIPLE. (W.O.P.)

To Prove: For all integers $n \geq 0$, $7^n - 1$ is divisible by 6.

Proof: Suppose, by way of contradiction, that there exists an integer $N \geq 0$ such that $7^N - 1$ is not divisible by 6.

Let $S = \{$ all integers t such that $t \geq 0$ and $7^t - 1$ is not divisible by 6. $\}$

By supposition, $N \geq 0$ and $7^N - 1$ is not divisible by 6, So, N is in Set S, and Set S is not empty.

Condition 1 of the W.O.P. is satisfied.

By definition of set S, for every integer t in Set S, $t \geq 0$. Condition 2 of the W.O.P. is satisfied.

By the Well-Ordering Principle, set S has a least element, m.

Note that $m \geq 0$ and $7^m - 1$ is not divisible by 6.

[We show that $m \geq 1$ by showing that $m \neq 0$.]

$7^0 - 1 = 0 = 0 \cdot 6$, so $7^0 - 1$ is divisible by 6, but $7^m - 1$ is not divisible by 6.

So, $m \neq 0$. $\therefore m \geq 1$ $\therefore m - 1 \geq 0$.

[We show that $7^{m-1}-1$ is divisible by 6.]

Suppose, by way of contradiction, that $7^{m-1}-1$ is not divisible by 6. Recall that $m-1 \geq 0$.

$\therefore$ Integer $m-1$ is in set S and $m-1 < m$, which contradicts the fact that m is the least element in S.

$\therefore$ $7^{m-1}-1$ is divisible by 6, by proof-by-contradiction.

[We show that $7^{m}-1$ is divisible by 6].

Since $7^{m-1}-1$ is divisible by 6, there exists an integer k such that $7^{m-1}-1 = 6k$.

$\therefore$ $7^{m-1} = 6k+1$ by R.O.A.

$\therefore$ $7^{m} = 42k+7 = 6 \times (7k) + 6 + 1$

$\therefore$ $7^{m}-1 = 6(7k+1)$ and $7k+1$ is an integer.

$\therefore$ $7^{m}-1$ is divisible by 6, which contradicts the fact that $7^{m}-1$ is not divisible by 6.

$\therefore$ For all integers $n \geq 0$, $7^{n}-1$ is divisible by 6, by proof-by-contradiction.

Q.E.D.